C^*-Algebras in Physics

through the Ideal Fermi Gas

BERKE UNAL

September 2026

Introduction

At the experimental level, a physical system is accessed through the physical quantities that can be probed, the preparation of the system, and its evolution in time. A physical theory represents physical quantities by observables and preparations by states, specifies the statistical predictions associated with each state and observable, and specifies their time evolution called dynamics.

In this manuscript, we present C^*-algebras as a natural mathematical framework for organizing these ingredients. We focus on quantum systems, which are usually formulated in terms of Hilbert spaces. We use the ideal Fermi gas as our main example to compare the usual Hilbert-space formulation with the C^*-algebraic formulation.

Contents

1 Hilbert-Space Formulation **3**
1.1 Observables as Self-Adjoint Operators on Hilbert Spaces 3
1.2 Positive Operators and Trace on Hilbert Spaces 3
1.3 States as Density Operators on Hilbert Spaces 4
1.4 Dynamics Associated to a Hamiltonian 4

2 C^*-Algebraic Formulation **4**
2.1 Observables as Self-Adjoint Elements of a C^*-Algebra 5
2.2 Positive Elements of C^*-Algebras 5
2.3 States on a C^*-Algebra 6
2.4 C^*-Dynamical Systems 7

3 Fermionic Systems **8**
3.1 Hilbert Space of Fermionic Systems 8
3.2 C*-Algebra of Fermionic Systems 9
3.3 Hamiltonian Dynamics of Noninteracting Fermionic Systems 10
3.4 C^*-Dynamical System of Noninteracting Fermionic Systems 10

4 Equilibrium States **11**
4.1 Grand Canonical Gibbs State 11
4.2 The Ideal Fermi Gas 12
4.3 Gibbs states without density operators 12
4.4 The Kubo–Martin–Schwinger Condition 14

5 Equilibrium States of the Ideal Fermi Gas **14**

1 Hilbert-Space Formulation

In the introduction, we stated that a mathematical description of a physical system should specify its observables, states, expectation values, and dynamics. In this chapter, we summarize how these elements are represented in the usual Hilbert-space formulation.

1.1 Observables as Self-Adjoint Operators on Hilbert Spaces

The usual Hilbert-space formulation of quantum mechanics starts with a complex Hilbert space $\mathcal{H}$, that is, a vector space over the complex field $\mathbb{C}$ equipped with an inner product

$$\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}$$

which is complete[1] with respect to the norm $\|f\| := \sqrt{\langle f, f \rangle}$.

Let $B(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$. For $A \in B(\mathcal{H})$, the adjoint[2] $A^* \in B(\mathcal{H})$ is the unique operator satisfying

$$\langle f, Ag \rangle = \langle A^* f, g \rangle \qquad f, g \in \mathcal{H}.$$

The bounded **observables** of the system are represented by self-adjoint operators, that is, operators $A \in B(\mathcal{H})$ satisfying

$$A = A^*.$$

1.2 Positive Operators and Trace on Hilbert Spaces

A positive operator $A \in B(\mathcal{H})$ is an operator satisfying

$$\langle f, Af \rangle \geq 0 \qquad f \in \mathcal{H}.$$

For a positive operator $A \in B(\mathcal{H})$, its trace $\mathrm{Tr}(A)$ is defined by

$$\mathrm{Tr}(A) := \sum_n \langle e_n, Ae_n \rangle,$$

where $(e_n)_n$ is an orthonormal basis of $\mathcal{H}$. The value of the sum, possibly infinite, is independent of the choice of orthonormal basis.

When $\mathcal{H}$ is infinite-dimensional, a positive operator need not be trace class, that is, its trace may be infinite. For example, the identity operator $I_{\mathcal{H}}$ is not trace-class:

$$\mathrm{Tr}(I_{\mathcal{H}}) = \infty.$$

The possible divergence of the trace will be crucial when we consider equilibrium states of the ideal Fermi gas.

---

[1] A normed vector space is called complete if every Cauchy sequence converges, that is, if $(f_n)_n$ satisfies $\|f_n - f_m\| \to 0$ as $n, m \to \infty$, then there exists $f \in \mathcal{H}$ such that $\|f_n - f\| \to 0$ as $n \to \infty$.

[2] In the physics literature, the adjoint is also commonly denoted by $A^\dagger$. The reason for the notation A^* will become apparent when we introduce C^*-algebras.

1.3 States as Density Operators on Hilbert Spaces

A **state** is given by a density operator $\rho \in B(\mathcal{H})$, that is, a positive operator satisfying

$$\mathrm{Tr}(\rho) = 1.$$

When the system is in the state given by a density operator ρ, the expectation value $\langle A \rangle_\rho$ of an observable A is given by

$$\langle A \rangle_\rho = \mathrm{Tr}(\rho A).$$

Physically, if the same preparation corresponding to ρ is repeated independently and the observable A is measured each time, the measurement outcomes are described by a probability distribution with expectation $\langle A \rangle_\rho$. By the law of large numbers, the empirical average of the outcomes converges to $\langle A \rangle_\rho$ as the number of repetitions increases.

1.4 Dynamics Associated to a Hamiltonian

The dynamics of the system is determined by a self-adjoint operator H on $\mathcal{H}$, called the Hamiltonian[3]. There are two equivalent descriptions of the dynamics: one can either let the states evolve or let the observables evolve.

In the Schrödinger picture, the observables are kept fixed and a density operator ρ evolves according to

$$\rho_t = e^{-itH} \rho e^{itH}.$$

In the Heisenberg picture, the state is kept fixed and an observable $A \in B(\mathcal{H})$ evolves according to

$$A_t = e^{itH} A e^{-itH}.$$

The two descriptions give the same expectation values $\langle A \rangle_\rho$ since

$$\mathrm{Tr}(\rho_t A) = \mathrm{Tr}(\rho A_t).$$

Thus the choice of picture changes only where the time dependence is placed, and not the physical predictions.

2 C^*-Algebraic Formulation

We now describe how observables, states, expectation values, and dynamics are formulated in the C^*-algebraic setting that generalizes the usual Hilbert-space formulation.

[3]While we will not develop the rigorous theory of unbounded operators here, we note that in many physical systems with infinite-dimensional Hilbert spaces, the corresponding Hamiltonian H is unbounded.

The theory of unbounded operators is substantially more delicate than that of bounded operators. In particular, self-adjointness requires careful treatment of operator domains: an operator H defined on all of $\mathcal{H}$ and satisfying $\langle Hf, g \rangle = \langle f, Hg \rangle$ for all $f, g \in \mathcal{H}$ is, by the Hellinger–Toeplitz theorem, necessarily bounded. Thus an unbounded self-adjoint operator cannot be defined on all of $\mathcal{H}$, and even the definition of the adjoint requires greater care.

Stone's theorem gives a one-to-one correspondence between possibly unbounded self-adjoint operators H and strongly continuous one-parameter unitary groups $(U_t)_{t \in \mathbb{R}}$ through $U_t = e^{-itH}$.

2.1 Observables as Self-Adjoint Elements of a C^*-Algebra

To construct a C^*-algebra, we start with a complex vector space $\mathcal{A}$ equipped with a complete norm $\|\cdot\|$. We then equip it with two additional operations.

The first is a multiplication of elements, $(A, B) \mapsto AB$, which is associative,

$$(AB)C = A(BC) \qquad A, B, C \in \mathcal{A}.$$

and bilinear, meaning that for every fixed $B \in \mathcal{A}$ the maps $A \mapsto AB$ and $A \mapsto BA$ are linear.

The second operation is a conjugate-linear involution, $A \mapsto A^*$, that is,

$$(\lambda A + \mu B)^* = \overline{\lambda} A^* + \overline{\mu} B^*, \qquad (A^*)^* = A, \qquad A, B \in \mathcal{A},\ \lambda, \mu \in \mathbb{C}$$

where $\overline{\lambda}$ denotes the complex conjugate of λ, and also satisfying

$$(AB)^* = B^* A^*, \qquad \|A^* A\| = \|A\|^2 \qquad A, B \in \mathcal{A}.$$

The **observables** of the system are represented by the self-adjoint elements of $\mathcal{A}$, that is, elements $A \in \mathcal{A}$ satisfying

$$A = A^*.$$

In general, a C^*-algebra need not have an identity element, that is, an element $\mathbf{1}$ satisfying

$$A\mathbf{1} = \mathbf{1}A = A \qquad A \in \mathcal{A}.$$

Here, however, we will work with C^*-algebras that have an identity element, called unital C^*-algebras.[4]

Observe that

$$\mathcal{A} = B(\mathcal{H})$$

is a unital C^*-algebra when equipped with the usual operator multiplication, operator norm, and adjoint, and that its self-adjoint elements are precisely the self-adjoint operators on $\mathcal{H}$.

2.2 Positive Elements of C^*-Algebras

An element $A \in \mathcal{A}$ is called positive, and we write $A \geq 0$, if there exists $B \in \mathcal{A}$ such that

$$A = B^* B.$$

In particular, every positive element is self-adjoint.[5]

For $\mathcal{A} = B(\mathcal{H})$, this notion agrees with positivity for operators on a Hilbert space. Indeed, if $A = B^* B$, then

$$\langle f, Af \rangle = \langle Bf, Bf \rangle = \|Bf\|^2 \geq 0 \qquad f \in \mathcal{H}.$$

Conversely, every positive operator $A \in B(\mathcal{H})$ can be written as $A = B^* B$ for some $B \in B(\mathcal{H})$, by taking $B = A^{1/2}$, the positive square root of A.

[4] Every C^*-algebra has an approximate identity, that is, there exists a net $(e_\lambda)_\lambda$ of elements such that $\|e_\lambda A - A\| \to 0$ and $\|A e_\lambda - A\| \to 0$ for every $A \in \mathcal{A}$.

[5] Equivalently, an element $A \in \mathcal{A}$ is positive if and only if it is self-adjoint and its spectrum

$$\sigma(A) := \{\lambda \in \mathbb{C} : A - \lambda \mathbf{1} \text{ is not invertible}\}$$

is contained in $[0, \infty)$. For an observable, the spectrum can be thought of as the set of possible values that the observable can take.

2.3 States on a C^*-Algebra

A **state** ω on a unital C^*-algebra $\mathcal{A}$ is a linear map

$$\omega : \mathcal{A} \longrightarrow \mathbb{C}$$

such that

$$A \geq 0 \quad \Longrightarrow \quad \omega(A) \geq 0 \qquad\qquad A \in \mathcal{A},$$

and

$$\omega(\mathbf{1}) = 1.$$

When the system is in the state ω, the expectation value $\langle A \rangle_\omega$ of an observable $A \in \mathcal{A}$ is given by

$$\langle A \rangle_\omega = \omega(A).$$

A state thus may be regarded as an assignment of expectation values to observables. The definition of state is motivated by the basic properties of expectation $\mathbb{E}$ in probability theory. $\mathbb{E}$ is linear,

$$X \geq 0 \quad \Longrightarrow \quad \mathbb{E}[X] \geq 0,$$

and

$$\mathbb{E}[1] = 1.$$

The essential difference is that, in the present setting, observables need not commute, that is, there may exist observables $A, B \in \mathcal{A}$ such that

$$AB \neq BA.$$

Thus, a unital C^*-algebra $\mathcal{A}$ together with a state ω may be viewed as a noncommutative probability theory, with observables playing the role of noncommutative random variables and ω playing the role of $\mathbb{E}$.[6]

The density-matrix description of states on a Hilbert space is contained in this formulation. If ρ is a density operator on $\mathcal{H}$, then

$$\omega_\rho : B(\mathcal{H}) \longrightarrow \mathbb{C}, \qquad A \longmapsto \mathrm{Tr}(\rho A),$$

defines a state on the unital C^*-algebra $B(\mathcal{H})$. The corresponding expectation value agrees with the one in the Hilbert-space formulation:

$$\langle A \rangle_{\omega_\rho} = \omega_\rho(A) = \mathrm{Tr}(\rho A) = \langle A \rangle_\rho.$$

In finite dimensions, every state on $B(\mathcal{H})$ arises from a density operator in this way. In infinite dimensions, however, not every state is representable by a density operator in a given Hilbert-space representation.

[6] If $\mathcal{A}$ is commutative, that is, if $AB = BA$ for all $A, B \in \mathcal{A}$, then this formulation genuinely recovers ordinary probability theory: the Gelfand representation identifies $\mathcal{A}$ with the C^*-algebra $C(X)$ of continuous functions on some compact Hausdorff topological space X. The self-adjoint elements correspond to real-valued continuous functions on X, and the Riesz–Markov–Kakutani representation theorem establishes a one-to-one correspondence between states ω on $C(X)$ and regular Borel probability measures μ on X, given by $\omega(f) = \int_X f \, d\mu$.

This will be crucial when we study the ideal Fermi gas. Its infinite-volume equilibrium state is naturally defined as a state on a C^*-algebra and, is not representable by a density operator on the usual Hilbert-space formulation (Fock space). Thus the C^*-algebraic formulation allows us to describe physically relevant states that are not available as density matrices in the usual Hilbert-space formulation.

2.4 C^*-Dynamical Systems

A $*$-automorphism of a C^*-algebra $\mathcal{A}$ is a bijective linear map

$$\alpha : \mathcal{A} \longrightarrow \mathcal{A}$$

satisfying

$$\alpha(AB) = \alpha(A)\alpha(B), \qquad \alpha(A^*) = \alpha(A)^* \qquad A, B \in \mathcal{A}.$$

A C^*-dynamical system $(\mathcal{A}, \alpha)$ is a C^*-algebra $\mathcal{A}$ together with a strongly continuous group of $*$-automorphisms α of $\mathcal{A}$, that is, a family $(\alpha_t)_{t\in\mathbb{R}}$ of $*$-automorphisms satisfying group property

$$\alpha_0 = \mathrm{id}_{\mathcal{A}}, \qquad \alpha_{t+s} = \alpha_t \circ \alpha_s \qquad t, s \in \mathbb{R},$$

and strong continuity

$$\|\alpha_t(A) - A\| \longrightarrow 0 \qquad \text{as } t \longrightarrow 0 \qquad A \in \mathcal{A}.$$

In the Heisenberg picture, the observable A at time t is

$$A_t = \alpha_t(A).$$

In the Schrödinger picture, the observable is kept fixed and a state ω evolves according to

$$\omega_t(A) := \omega(\alpha_t(A)) \qquad A \in \mathcal{A}.$$

Observe that

$$\langle A \rangle_{\omega_t} = \omega_t(A) = \omega(A_t) = \langle A_t \rangle_\omega$$

so evolving the state or evolving the observable gives the same expectation values.

The usual Heisenberg evolution on a Hilbert space is a special case. If a self-adjoint Hamiltonian H on $\mathcal{H}$, then

$$\alpha_t(A) = e^{itH} A e^{-itH} \qquad A \in B(\mathcal{H})$$

defines a strongly continuous group of $*$-automorphisms of $B(\mathcal{H})$. Thus the C^*-algebraic description recovers the usual Heisenberg picture whenever the dynamics is implemented by a Hamiltonian on a Hilbert space. The same is true for the Schrödinger picture.

Summary

	Hilbert-space formulation	**C^*-algebraic formulation**
Observables	self-adjoint operators on $\mathcal{H}$	self-adjoint elements of $\mathcal{A}$
States	density operators ρ on $\mathcal{H}$	states ω on $\mathcal{A}$
Expectations	$\langle A\rangle_\rho = \mathrm{Tr}(\rho A)$	$\langle A\rangle_\omega = \omega(A)$
Dynamics	Hamiltonian H	strongly continuous group of $*$-automorphisms $(\alpha_t)_{t\in\mathbb{R}}$
	$A_t = e^{itH} A e^{-itH}$	$A_t = \alpha_t(A)$
	$\rho_t = e^{-itH}\rho e^{itH}$	$\omega_t(A) = \omega(\alpha_t(A))$

3 Fermionic Systems

In this section, we describe both the Hilbert-space and the C^*-algebraic formulations of fermionic systems. We first introduce the fermionic Fock space and the corresponding C^*-algebra of observables.

We then study noninteracting dynamics first through a Hamiltonian in the Hilbert-space formulation, and then through a strongly continuous one-parameter group of $*$-automorphisms in the C^*-algebraic formulation.

Some of the constructions used in this section require additional background that we will not develop in detail here. The reader may consult standard references for the relevant definitions when needed.

3.1 Hilbert Space of Fermionic Systems

Let $\mathcal{H}$ denote the one-particle Hilbert space. $n \in \mathbb{N}$ identical fermions is described by the antisymmetric n-fold tensor product $\bigwedge^n \mathcal{H}$, reflecting the antisymmetry of fermions under particle exchange. Allowing the particle number to vary leads to the fermionic Fock space $\mathcal{F}_-(\mathcal{H})$, that is the Hilbert-space direct sum of the antisymmetric tensor powers of $\mathcal{H}$

$$\mathcal{F}_-(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \bigwedge^n \mathcal{H}.$$

This is the Hilbert space of fermionic systems.

For $f \in \mathcal{H}$, the annihilation operator $a_f \in \mathcal{B}(\mathcal{F}_-(\mathcal{H}))$ is defined by

$$a_f \Psi := \bigoplus_{n=1}^{\infty} \iota_f^{(n)} \Psi^{(n)}$$

where,

$$\iota_f^{(n)} : \bigwedge^n \mathcal{H} \longrightarrow \bigwedge^{n-1} \mathcal{H}$$

denotes contraction by f.[7]

The Hilbert space adjoint a_f^* of a_f is called the creation operator and acts by exterior multiplication,

$$a_f^* \Psi = 0 \oplus \bigoplus_{n=0}^{\infty} \big(f \wedge \Psi^{(n)}\big).$$

The creation and annihilation operators satisfy the canonical anticommutation relations,

$$\{a_f, a_g\} = 0, \qquad \{a_f^*, a_g^*\} = 0, \qquad \{a_f, a_g^*\} = \langle f, g\rangle \, I_{\mathcal{F}_-(\mathcal{H})}.$$

In particular,

$$(a_f^*)^2 = 0,$$

which is the Pauli exclusion principle.

3.2 C*-Algebra of Fermionic Systems

The canonical anticommutation relations C^*-algebra over $\mathcal{H}$ is the unital C^*-subalgebra of $\mathcal{B}(\mathcal{F}_-(\mathcal{H}))$ generated by the annihilation operators,

$$\mathrm{CAR}(\mathcal{H}) := C^*(a_f \in \mathcal{B}(\mathcal{F}_-(\mathcal{H})) : f \in \mathcal{H}).$$

This is the C^*-algebra of the fermionic systems.[8]

If $\mathcal{H}$ is infinite-dimensional, then $\mathrm{CAR}(\mathcal{H})$ is strictly smaller than $\mathcal{B}(\mathcal{F}_-(\mathcal{H}))$

$$\mathrm{CAR}(\mathcal{H}) \subsetneq \mathcal{B}(\mathcal{F}_-(\mathcal{H})).$$

Thus, the observables of $\mathrm{CAR}(\mathcal{H})$ do not coincide with the observables of the Hilbert-space $\mathcal{F}_-(\mathcal{H})$ formulation.

One may ask whether there is still a way to use the Hilbert-space formulation, using a Hilbert space $\widetilde{\mathcal{H}}$ different than $\mathcal{F}_-(\mathcal{H})$: Could $\mathrm{CAR}(\mathcal{H})$ nevertheless be of the form

$$\mathrm{CAR}(\mathcal{H}) \cong \mathcal{B}(\widetilde{\mathcal{H}})$$

for some other Hilbert space $\widetilde{\mathcal{H}}$?

The answer is no: If $\widetilde{\mathcal{H}}$ is finite-dimensional, then $\mathcal{B}(\widetilde{\mathcal{H}})$ is finite-dimensional, whereas $\mathrm{CAR}(\mathcal{H})$ is not. If $\widetilde{\mathcal{H}}$ is infinite-dimensional, then $\mathcal{B}(\widetilde{\mathcal{H}})$ has no tracial state, while $\mathrm{CAR}(\mathcal{H})$ does. Thus, for infinite-dimensional $\mathcal{H}$,

$$\mathrm{CAR}(\mathcal{H}) \ncong \mathcal{B}(\widetilde{\mathcal{H}})$$

for any Hilbert space $\widetilde{\mathcal{H}}$. The study of $\mathrm{CAR}(\mathcal{H})$ therefore requires genuine C^*-algebraic tools, rather than the usual Hilbert-space ones.

[7] The map $\mathcal{H} \to \mathcal{B}(\mathcal{F}_-(\mathcal{H}))$, $f \mapsto a_f$, is a conjugate-linear isometry, that is, $a_{\lambda f + \mu g} = \overline{\lambda} a_f + \overline{\mu} a_g$ and $\|a_f\| = \|f\|$.

[8] The algebra $\mathrm{CAR}(\mathcal{H})$ can also be defined abstractly, independently of the Fock-space construction, as the universal C^*-algebra generated by elements satisfying the canonical anticommutation relations. Technically speaking, the construction given here realizes this abstract C^*-algebra through its faithful Fock representation. This concrete realization is useful for directly comparing the Hilbert-space formulation with the C^*-algebraic formulation.

3.3 Hamiltonian Dynamics of Noninteracting Fermionic Systems

We now consider noninteracting fermionic systems. By noninteracting, we mean that the particles do not interact with one another, so that the dynamics of each particle is determined solely by the same one-particle Hamiltonian H on $\mathcal{H}$.

In the usual Hilbert-space formulation, in order to describe the noninteracting dynamics of the full fermionic system, one passes from a one-particle Hamiltonian H on $\mathcal{H}$ to a Hamiltonian $d\Gamma(H)$ on $\mathcal{F}_-(\mathcal{H})$. The process of passing from one-particle operators to operators on Fock space is called second quantization and is carried out as follows:

The n-particle Hamiltonian $H^{(n)}$ on $\bigwedge^n \mathcal{H}$ is

$$H^{(n)} := \sum_{j=1}^{n} I_{\mathcal{H}}^{\otimes(j-1)} \otimes H \otimes I_{\mathcal{H}}^{\otimes(n-j)},$$

restricted to the antisymmetric n-particle space, and set $H^{(0)} := 0$.

The second quantization of H is the self-adjoint operator $d\Gamma(H)$ on $\mathcal{F}_-(\mathcal{H})$ given by

$$d\Gamma(H) := \bigoplus_{n=0}^{\infty} H^{(n)}.$$

The corresponding dynamics on $\mathcal{B}(\mathcal{F}_-(\mathcal{H}))$, in the Heisenberg picture, is given by

$$A \longmapsto e^{itd\Gamma(H)} A e^{-itd\Gamma(H)}.$$

For an unbounded one-particle Hamiltonian H, the rigorous construction of $d\Gamma(H)$ is notoriously more difficult and involves specifying the domain of the direct sum and taking the appropriate self-adjoint closures.

3.4 C^*-Dynamical System of Noninteracting Fermionic Systems

In the C^*-algebraic formulation, however, the dynamics on $\mathrm{CAR}(\mathcal{H})$ can be defined directly even when H is unbounded, without first constructing a Hamiltonian on $\mathcal{F}_-(\mathcal{H})$.

The key observation is that

$$e^{itd\Gamma(H)} a_f e^{-itd\Gamma(H)} = a_{e^{itH} f} \qquad f \in \mathcal{H}.$$

Since $\mathrm{CAR}(\mathcal{H})$ is generated by the annihilation operators a_f, it follows that the time evolution maps every observable in $\mathrm{CAR}(\mathcal{H})$ to another observable in $\mathrm{CAR}(\mathcal{H})$. This suggests focusing on the dynamics internal to $\mathrm{CAR}(\mathcal{H})$, rather than describing the dynamics through a Hamiltonian on the Hilbert space $\mathcal{F}_-(\mathcal{H})$.

Accordingly, we define the dynamics directly on the generators of $\mathrm{CAR}(\mathcal{H})$ by

$$\alpha_t^H(a_f) := a_{e^{itH} f} \qquad f \in \mathcal{H}.$$

Since e^{itH} is unitary, the operators $a_{e^{itH} f}$ satisfy the same canonical anticommutation relations as the a_f. Therefore this prescription extends uniquely to a $*$-automorphism

$$\alpha_t^H : \mathrm{CAR}(\mathcal{H}) \longrightarrow \mathrm{CAR}(\mathcal{H}).$$

One can thus show that the one-parameter unitary group $(e^{itH})_{t\in\mathbb{R}}$ induces a strongly continuous group of $*$-automorphisms $(\alpha_t^H)_{t\in\mathbb{R}}$ of $\mathrm{CAR}(\mathcal{H})$. Hence, we obtain the C^*-dynamical system

$$\big(\mathrm{CAR}(\mathcal{H}), \alpha^H\big).$$

This is the C^*-dynamical system of the noninteracting fermionic systems.

4 Equilibrium States

We begin with the equilibrium states of noninteracting fermionic quantum systems and use the ideal Fermi gas, that is, a system of indistinguishable fermions moving in the continuum without interparticle interactions, as an example to test these constructions.

4.1 Grand Canonical Gibbs State

The number operator N on $\mathcal{F}_-(\mathcal{H})$ is the self-adjoint operator defined by

$$N\Psi := 0 \oplus \bigoplus_{n=1}^{\infty} n\Psi^{(n)},$$

$$\mathcal{D}(N) := \big\{\Psi \in \mathcal{F}_-(\mathcal{H}) : \sum_{n=0}^{\infty} n^2 \|\Psi^{(n)}\|^2 < \infty\big\}.$$

It is the second quantization of the identity operator on $\mathcal{H}$

$$N = d\Gamma(I_{\mathcal{H}}).$$

For a chemical potential $\mu \in \mathbb{R}$, define the shifted one-particle Hamiltonian

$$H_\mu := H - \mu I_{\mathcal{H}}.$$

Its second quantization $d\Gamma(H_\mu)$[9] is called grand-canonical Hamiltonian on $\mathcal{F}_-(\mathcal{H})$

$$d\Gamma(H_\mu) = d\Gamma(H) - \mu N.$$

For inverse temperature $\beta \in \mathbb{R}$, the grand-canonical partition function $Z_{\beta,\mu}$ is defined by

$$Z_{\beta,\mu} := \mathrm{Tr}\left(e^{-\beta d\Gamma(H_\mu)}\right).$$

If $e^{-\beta d\Gamma(H_\mu)}$ is trace class[10], that is, $Z_{\beta,\mu} < \infty$, then the following expression

$$\rho_{\beta,\mu} := \frac{e^{-\beta d\Gamma(H_\mu)}}{Z_{\beta,\mu}}$$

defines a density operator $\rho_{\beta,\mu}$ on $\mathcal{F}_-(\mathcal{H})$, and the corresponding state is the grand-canonical Gibbs state.

[9] It follows from the previous section, by replacing H with H_μ, that the dynamics associated with $d\Gamma(H_\mu)$ induces the strongly continuous group $\alpha^{H_\mu} = \big(\alpha_t^{H_\mu}\big)_{t\in\mathbb{R}}$ of $*$-automorphisms of $\mathrm{CAR}(\mathcal{H})$, determined by $\alpha_t^{H_\mu}(a_f) := a_{e^{itH_\mu}f}$.

Since $f \mapsto a_f$ is conjugate-linear, $\alpha_t^{H_\mu}(a_f) = e^{it\mu}\alpha_t^H(a_f)$. Thus the chemical potential modifies the noninteracting dynamics on the generators by the phase rotation $e^{it\mu}$.

[10] One can show that $e^{-\beta H}$ is trace class on $\mathcal{H}$ if and only if $e^{-\beta d\Gamma(H_\mu)}$ is trace class on $\mathcal{F}_-(\mathcal{H})$ for every $\mu \in \mathbb{R}$.

4.2 The Ideal Fermi Gas

For the ideal Fermi gas in infinite volume, the one-particle Hilbert space $\mathcal{H}$ is the space of square-integrable functions on $\mathbb{R}^d$,

$$\mathcal{H} = L^2(\mathbb{R}^d).^{11}$$

For a free particle of mass m, the one-particle Hamiltonian H is

$$H := -\frac{1}{2m}\Delta,$$

where Δ denotes the Laplace operator on $\mathbb{R}^d$.

The thing is that, for any $\beta, \mu \in \mathbb{R}$, the operator $e^{-\beta d\Gamma(H_\mu)}$ is not trace class on $\mathcal{F}_-(\mathcal{H})$, and therefore

$$Z_{\beta,\mu} = \infty,$$

hence the expression

$$\rho_{\beta,\mu} = \frac{e^{-\beta d\Gamma(H_\mu)}}{Z_{\beta,\mu}}$$

does not define a density operator on $\mathcal{F}_-(\mathcal{H})$ and therefore does not correspond to a state of the system.

The usual Hilbert-space formulation of the equilibrium state in terms of a density operator on the fermionic Fock space is therefore not appropriate.

4.3 Gibbs states without density operators

In the preceding subsection, we saw that the grand-canonical Gibbs density operator does not exist for the ideal Fermi gas in infinite volume. We therefore seek a characterization of thermal equilibrium which can be formulated even in infinite volume.

We return to the finite-dimensional setting, where the Gibbs state is well-defined. Suppose that H is a self-adjoint Hamiltonian on a finite-dimensional Hilbert space $\mathcal{H}$.

Recall that, for $\beta > 0$, the Gibbs density operator is given by

$$\rho_\beta := \frac{e^{-\beta H}}{Z_\beta},$$

where $Z_\beta = \mathrm{Tr}(e^{-\beta H})$. The corresponding Gibbs state is

$$\omega_\beta(A) = \frac{\mathrm{Tr}(e^{-\beta H}A)}{Z_\beta} \qquad A \in B(\mathcal{H}).$$

and the dynamics is

$$\alpha_t^H(A) = e^{itH}Ae^{-itH}.$$

[11] The precise definition of the Hilbert space $L^2(\mathbb{R}^d)$ is as follows. Let λ denote Lebesgue measure on $\mathbb{R}^d$. It consists of Lebesgue measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that $\int_{\mathbb{R}^d} |f|^2 \, d\lambda < \infty$, modulo equality λ-almost everywhere, with pointwise vector-space operations and inner product $\langle f, g\rangle = \int_{\mathbb{R}^d} \overline{f}\, g \, d\lambda$. Completeness follows from the Riesz–Fischer theorem.

Since H is bounded in the present finite-dimensional setting, the operator exponential

$$e^{izH} = \sum_{n=0}^{\infty} \frac{(izH)^n}{n!}$$

converges in operator norm for every $z \in \mathbb{C}$. Thus, the real-time dynamics admits an extension to complex parameters, given by

$$\alpha_z^H(A) := e^{izH} A e^{-izH}.$$

In particular, at $z = i\beta$, we have

$$\alpha_{i\beta}^H(B) = e^{-\beta H} B e^{\beta H}.$$

Using the cyclicity of the trace, we obtain

$$\begin{aligned} \operatorname{Tr}\left(e^{-\beta H} A \alpha_{i\beta}^H(B)\right) &= \operatorname{Tr}\left(e^{-\beta H} A e^{-\beta H} B e^{\beta H}\right) \\ &= \operatorname{Tr}\left(e^{-\beta H} B A\right). \end{aligned}$$

Dividing both sides by Z_β, we obtain

$$\omega_\beta\left(A \alpha_{i\beta}^H(B)\right) = \omega_\beta(BA).$$

Interestingly, this condition is also sufficient to determine the Gibbs state. Suppose that a state ω on $B(\mathcal{H})$ satisfies the above condition for all $A, B \in B(\mathcal{H})$. Since $\mathcal{H}$ is finite-dimensional, there exists a density operator ρ such that $\omega(A) = \operatorname{Tr}(\rho A)$. Using the cyclicity of the trace, we obtain

$$\operatorname{Tr}\left(\left(e^{\beta H} \rho A e^{-\beta H} - A\rho\right) B\right) = 0 \qquad A, B \in B(\mathcal{H}).$$

Since this holds for every $B \in B(\mathcal{H})$, the nondegeneracy of the trace pairing implies

$$e^{\beta H} \rho A = A \rho e^{\beta H} \qquad A \in B(\mathcal{H}).$$

Taking $A = I_{\mathcal{H}}$, we see that $e^{\beta H}\rho$ commutes with every operator in $B(\mathcal{H})$, and therefore a multiple of the identity

$$e^{\beta H} \rho = c I_{\mathcal{H}}$$

for some $c \in \mathbb{C}$. Since $\operatorname{Tr}(\rho) = 1$, we conclude that

$$\rho = \frac{e^{-\beta H}}{\operatorname{Tr}(e^{-\beta H})}.$$

Thus, in finite dimensions, the identity characterizes the Gibbs state.

The important observation is that this characterization involves *only* the states and the dynamics. This motivates the Kubo–Martin–Schwinger condition, which allows us to characterize thermal equilibrium states without referring to a density operator, a partition function, or even a Hilbert space.

4.4 The Kubo–Martin–Schwinger Condition

We now formulate the identity obtained in the preceding subsection for a general C^*-dynamical system $(\mathcal{A}, \alpha)$. Recall that the dynamics $(\alpha_t)_{t\in\mathbb{R}}$ is defined for real parameters. Unlike in the finite-dimensional setting, the map $t \mapsto \alpha_t(B)$ need not admit an analytic[12] extension to complex parameters for every $B \in \mathcal{A}$. We therefore restrict our attention to elements for which such an extension exists.

An element $B \in \mathcal{A}$ is called **entire analytic** for α if the map

$$\mathbb{R} \longrightarrow \mathcal{A}, \qquad t \longmapsto \alpha_t(B)$$

admits an analytic extension to the entire complex plane

$$\mathbb{C} \longrightarrow \mathcal{A}, \qquad z \longmapsto \alpha_z(B).$$

Let $\beta > 0$. A state ω on a C^*-dynamical system $(\mathcal{A}, \alpha)$ is called a β-KMS state if for every $A \in \mathcal{A}$ and every entire analytic element $B \in \mathcal{A}$

$$\omega\left(A\alpha_{i\beta}(B)\right) = \omega(BA).$$

This is the Kubo–Martin–Schwinger (KMS) condition.[13]

In the finite-dimensional setting considered in the preceding subsection, every element of $B(\mathcal{H})$ is entire analytic for the dynamics α^H. Thus, the KMS condition reduces precisely to the identity that characterizes the Gibbs state.

The important point is that the KMS condition is formulated entirely in terms of a C^*-algebra, a state, and its dynamics.

5 Equilibrium States of the Ideal Fermi Gas

Returning to the ideal Fermi gas in infinite volume, consider the C^*-dynamical system

$$\big(\mathrm{CAR}(L^2(\mathbb{R}^d)), \alpha^{H_\mu}\big)$$

introduced in the preceding sections. Although the grand-canonical Gibbs density operator does not exist on the fermionic Fock space, the KMS condition is still well-defined for this system. We are therefore led to ask whether there exists a state on $\mathrm{CAR}(L^2(\mathbb{R}^d))$ satisfying the β-KMS condition with respect to α^{H_μ}.

To be continued.

[12] A function $f : \mathbb{C} \to \mathcal{A}$ is called analytic if, for every $z \in \mathbb{C}$, the limit $\lim_{h\to 0}\big(f(z+h) - f(z)\big)/h$ exists with respect to the norm of $\mathcal{A}$. Equivalently, for every $z_0 \in \mathbb{C}$, it admits a power-series expansion $f(z) = \sum_{n=0}^{\infty} A_n (z - z_0)^n$, where $A_n \in \mathcal{A}$, converging in norm for z in a neighborhood of z_0. The entire analytic elements form a norm-dense $*$-subalgebra of $\mathcal{A}$.

[13] Equivalently, the KMS condition can be formulated in terms of functions analytic on a strip in the complex plane. The two formulations are equivalent for strongly continuous one-parameter groups of $*$-automorphisms.