CMSP / Operator Algebras as Physical Systems

Meeting 4:

Motivating C*-algebras

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Section 1

Operators on Hilbert Spaces

The Algebra of Bounded Operators

Let $\mathcal{H}$ be a nonzero complex Hilbert space. As in Meeting 3, the inner product is linear in its second variable.

$$\mathcal{B}(\mathcal{H}) = \{T : \mathcal{H} \to \mathcal{H} : T \text{ is bounded and linear}\}, \qquad (ST)x = S(Tx).$$

Composition is associative and bilinear, and its identity is I. Moreover,

$$\|STx\| \leq \|S\| \, \|Tx\| \leq \|S\| \, \|T\| \, \|x\| \quad \Longrightarrow \quad \|ST\| \leq \|S\| \, \|T\| \, .$$

What is already available

Meeting 3 gives completeness of $\mathcal{B}(\mathcal{H})$ in the operator norm and the existence of the adjoint T^*. We now check how the adjoint interacts with multiplication and the norm.

The Adjoint Reverses Products

The defining identity is $\langle Tx, y\rangle = \langle x, T^*y\rangle$. It gives

$$(\alpha S + \beta T)^* = \overline{\alpha} S^* + \overline{\beta} T^*,$$
$$(ST)^* = T^* S^*, \qquad (T^*)^* = T, \qquad I^* = I.$$

For the product identity, compute

$$\langle STx, y\rangle = \langle Tx, S^*y\rangle = \langle x, T^*S^*y\rangle .$$

Uniqueness of the adjoint gives $(ST)^* = T^*S^*$.

Conjugate symmetry also gives $\langle T^*x, y\rangle = \langle x, Ty\rangle$, so $(T^*)^* = T$. The scalar rule follows from conjugate-linearity in the first variable.

Taking the Adjoint Preserves the Norm

Cauchy–Schwarz gives, for every $z \in \mathcal{H}$,

$$\|z\| = \sup_{\|x\| \leq 1} |\langle x, z \rangle| .$$

For $z \neq 0$, equality is obtained by taking $x = z/\|z\|$. Therefore

$$\begin{aligned} \|T^* y\| &= \sup_{\|x\| \leq 1} |\langle x, T^* y \rangle| \\ &= \sup_{\|x\| \leq 1} |\langle Tx, y \rangle| \leq \|T\| \, \|y\| . \end{aligned}$$

Thus $\|T^*\| \leq \|T\|$. Apply this to T^* and use $(T^*)^* = T$ to obtain

$$\boxed{\|T^*\| = \|T\| .}$$

The Identity $\|T^*T\| = \|T\|^2$

Submultiplicativity and the preceding result give the first inequality:

$$\|T^*T\| \leq \|T^*\|\,\|T\| = \|T\|^2.$$

For the reverse inequality, let $\|x\| \leq 1$. Then

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle x, T^*Tx\rangle \leq \|x\|\,\|T^*Tx\| \leq \|T^*T\|.$$

Taking the supremum over these x gives $\|T\|^2 \leq \|T^*T\|$.

Conclusion

$$\boxed{\|T^*T\| = \|T\|^2 \qquad (T \in \mathcal{B}(\mathcal{H})).}$$

Completeness, Multiplication, and Adjoint

The space $\mathcal{B}(\mathcal{H})$ has the following structure:

- It is a complete complex normed vector space.
- It has an associative, bilinear product with $\|ST\| \leq \|S\| \, \|T\|$.
- The adjoint is conjugate-linear, reverses products, and satisfies $(T^*)^* = T$.
- Its norm and adjoint satisfy $\|T^*T\| = \|T\|^2$.

It also has a multiplicative identity, with $\|I\| = 1$.

Section 2

Finite Groups and Their Group Algebras

From a Group Action to Linear Combinations of Operators

Let G be a finite group. We use ordinary unitary representations,

$$U : G \to \mathcal{U}(\mathcal{H}), \qquad U(st) = U(s)U(t), \qquad U(s^{-1}) = U(s)^*.$$

Besides the individual operators $U(s)$, consider, for complex coefficients,

$$A = \sum_{s \in G} a_s U(s), \qquad B = \sum_{t \in G} b_t U(t).$$

Products and adjoints are again linear combinations of the same operators:

$$AB = \sum_{s,t \in G} a_s b_t U(st),$$
$$A^* = \sum_{s \in G} \overline{a_s}\, U(s^{-1}).$$

We first define these operations without choosing a representation.

The Group Algebra $\mathbb{C}[G]$

For $s \in G$, let $\delta_s : G \to \mathbb{C}$ be 1 at s and 0 elsewhere. Every function $a : G \to \mathbb{C}$ has a unique expansion

$$a = \sum_{s \in G} a_s \delta_s, \qquad a_s = a(s).$$

The complex vector space of these functions is denoted by $\mathbb{C}[G]$. Define its multiplication by

$$\delta_s \delta_t = \delta_{st}$$

and extend bilinearly:

$$\left(\sum_s a_s \delta_s \right) \left(\sum_t b_t \delta_t \right) = \sum_{s,t} a_s b_t \delta_{st}.$$

Associativity comes from $(st)r = s(tr)$, and the identity is δ_e.

The Product in Terms of Coefficients

To find the coefficient of δ_r in ab, collect the pairs with $st = r$. For each s, this forces $t = s^{-1}r$.

$$\boxed{(ab)(r) = \sum_{s \in G} a(s)b(s^{-1}r).}$$

This product is called *convolution*. We will also write it as $a * b$.

Example: $G = \mathbb{Z}_2 = \{e, p\}$, where $p^2 = e$

$$(\alpha\delta_e + \beta\delta_p)(\gamma\delta_e + \eta\delta_p) = (\alpha\gamma + \beta\eta)\delta_e + (\alpha\eta + \beta\gamma)\delta_p.$$

The Involution on the Group Algebra

Define

$$\delta_s^* = \delta_{s^{-1}}, \qquad \left(\sum_s a_s \delta_s\right)^* = \sum_s \overline{a_s}\, \delta_{s^{-1}}.$$

Equivalently,

$$a^*(r) = \overline{a(r^{-1})}.$$

It is conjugate-linear, and $(a^*)^* = a$. On basis elements,

$$(\delta_s \delta_t)^* = \delta_{(st)^{-1}} = \delta_{t^{-1}s^{-1}} = \delta_t^* \delta_s^*.$$

Hence $(ab)^* = b^* a^*$ for all $a, b \in \mathbb{C}[G]$.

A Group Representation Extends to the Group Algebra

Given $U : G \to \mathcal{U}(\mathcal{H})$, define the linear map

$$\pi_U : \mathbb{C}[G] \to \mathcal{B}(\mathcal{H}), \qquad \pi_U(a) = \sum_s a_s U(s).$$

The group law gives

$$\begin{aligned}\pi_U(ab) &= \sum_{s,t} a_s b_t U(st) \\ &= \left(\sum_s a_s U(s)\right)\left(\sum_t b_t U(t)\right) = \pi_U(a)\pi_U(b).\end{aligned}$$

Also,

$$\pi_U(a^*) = \sum_s \overline{a_s} U(s)^* = \pi_U(a)^*, \qquad \pi_U(\delta_e) = I.$$

Thus π_U preserves products and adjoints, as well as linear combinations.

One Representation May Not Detect Every Element

Since each $U(s)$ is unitary,

$$\|\pi_U(a)\| \leq \sum_s |a_s| \, \|U(s)\| = \sum_s |a_s| =: \|a\|_1 .$$

So $a \mapsto \|\pi_U(a)\|$ is finite, homogeneous, and subadditive. But it may vanish at a nonzero element.

The trivial representation of $\mathbb{Z}_2$

Take $U(e) = U(p) = 1$ on $\mathbb{C}$. Then

$$\pi_U(\delta_e - \delta_p) = 1 - 1 = 0, \qquad \delta_e - \delta_p \neq 0.$$

To obtain a norm, we need enough representations to distinguish nonzero coefficient functions.

The Left Regular Representation of a Finite Group

Use the Hilbert space $\ell^2(G)$ with orthonormal basis $\{\delta_t : t \in G\}$. Define

$$\lambda(s)\delta_t = \delta_{st}.$$

Each $\lambda(s)$ permutes this basis, so it is unitary. Moreover,

$$\lambda(s)\lambda(t)\delta_r = \delta_{str} = \lambda(st)\delta_r.$$

Thus λ is a unitary representation. In function notation,

$$(\lambda(s)\xi)(r) = \xi(s^{-1}r).$$

Its linear extension is $\lambda(a) = \sum_s a_s \lambda(s)$.

The Regular Representation Detects Every Coefficient

Apply $\lambda(a)$ to the basis vector at the identity:

$$\lambda(a)\delta_e = \sum_s a_s \lambda(s)\delta_e = \sum_s a_s \delta_s = a.$$

Consequently,

$$\lambda(a) = 0 \quad \Longrightarrow \quad a = 0.$$

Faithfulness

A representation of an algebra is called *faithful* when it is injective. The map $a \mapsto \lambda(a)$ is faithful.

Here the same function a is viewed first as an algebra element and then as a vector of $\ell^2(G)$.

The Full Norm for a Finite Group

Define

$$\boxed{\|a\|_{\mathrm{u}} = \sup_{U} \|\pi_U(a)\|,}$$

where U ranges over all unitary representations of G on nonzero complex Hilbert spaces.

$$0 \leq \|a\|_{\mathrm{u}} \leq \|a\|_1 < \infty.$$

The operator norm gives

$$\|\alpha a\|_{\mathrm{u}} = |\alpha| \, \|a\|_{\mathrm{u}}, \qquad \|a+b\|_{\mathrm{u}} \leq \|a\|_{\mathrm{u}} + \|b\|_{\mathrm{u}}.$$

Finally, λ is one of the representations in the supremum, so

$$\|a\|_{\mathrm{u}} = 0 \implies \|\lambda(a)\| = 0 \implies a = 0.$$

This is a norm, called the *full* or *universal* norm.

The Full Norm Respects Products and Adjoints

For every U,

$$\|\pi_U(ab)\| = \|\pi_U(a)\pi_U(b)\| \leq \|\pi_U(a)\|\,\|\pi_U(b)\| \leq \|a\|_{\mathrm{u}}\,\|b\|_{\mathrm{u}}\,.$$

Taking the supremum gives

$$\boxed{\|ab\|_{\mathrm{u}} \leq \|a\|_{\mathrm{u}}\,\|b\|_{\mathrm{u}}\,.}$$

Likewise,

$$\|a^*\|_{\mathrm{u}} = \sup_U \|\pi_U(a)^*\| = \sup_U \|\pi_U(a)\| = \|a\|_{\mathrm{u}}\,.$$

These are consequences of the corresponding operator identities, not additional assumptions on $\mathbb{C}[G]$.

The Norm-Square Identity for a Finite Group

For each unitary representation U, the operator identity proved in Section 1 gives

$$\|\pi_U(a^*a)\| = \|\pi_U(a)^*\pi_U(a)\| = \|\pi_U(a)\|^2 .$$

Therefore

$$\begin{aligned} \|a^*a\|_{\mathrm{u}} &= \sup_U \|\pi_U(a)\|^2 \\ &= \left(\sup_U \|\pi_U(a)\|\right)^2 = \|a\|_{\mathrm{u}}^2 . \end{aligned}$$

Also, $\dim \mathbb{C}[G] = |G| < \infty$, so it is complete in this norm.

Computing the Full Norm for $\mathbb{Z}_2$

Let $a = \alpha\delta_e + \beta\delta_p$. In any unitary representation, $V = U(p)$ satisfies $V^2 = I$ and $V^* = V$. The operators

$$P_+ = \frac{I+V}{2}, \qquad P_- = \frac{I-V}{2}$$

are orthogonal projections, with $P_+ + P_- = I$ and $P_+P_- = 0$. Thus

$$\pi_U(a) = (\alpha+\beta)P_+ + (\alpha-\beta)P_-.$$

Orthogonality gives, for every x,

$$\|\pi_U(a)x\|^2 \le \max\{|\alpha+\beta|^2, |\alpha-\beta|^2\}\,\|x\|^2.$$

The one-dimensional representations $U(p) = 1$ and $U(p) = -1$ attain the two scalar values. Therefore

$$\boxed{\|\alpha\delta_e + \beta\delta_p\|_{\mathrm{u}} = \max\{|\alpha+\beta|, |\alpha-\beta|\}.}$$

Section 3

Locally Compact Groups

From Finite Groups to Locally Compact Groups

Now let G be a locally compact Hausdorff topological group. Its multiplication and inversion are continuous.

Examples include discrete groups, $(\mathbb{R}, +)$, the circle group $U(1)$, and $GL_n(\mathbb{R})$.

Continuity of representations

As in Meeting 3, a unitary representation U is *strongly continuous* when $s \mapsto U(s)\xi$ is norm-continuous for every $\xi \in \mathcal{H}$. We require this from now on.

Finite sums will be replaced by integrals. To make this replacement, we need a measure that respects the group law.

We do not assume that $s \mapsto U(s)$ is continuous in operator norm.

Haar Measure Replaces Counting Measure

Haar measure existance

There is a nonzero regular Borel measure μ on G that is finite on compact sets, positive on nonempty open sets, and left-invariant:

$$\mu(sE) = \mu(E).$$

It is unique up to multiplication by a positive constant.

Fix such a measure and write ds for $d\mu(s)$. Left invariance means

$$\int_G F(st)\,dt = \int_G F(t)\,dt \qquad (s \in G).$$

Counting measure works for discrete groups; Lebesgue measure works for $(\mathbb{R}, +)$. On $U(1)$ one can use $d\theta/(2\pi)$.

Analytic background: Williams, §1.3; Folland, Chapter 2.

Continuous Functions with Compact Support

We begin with

$$C_c(G) = \{f : G \to \mathbb{C} : f \text{ is continuous and } \operatorname{supp} f \text{ is compact}\},$$

where $\operatorname{supp} f = \overline{\{s : f(s) \neq 0\}}$. For $f \in C_c(G)$, both quantities

$$\|f\|_1 = \int_G |f(s)|\, ds, \qquad \|f\|_2 = \left(\int_G |f(s)|^2\, ds\right)^{1/2}$$

are finite.

Comparison with the finite case

For a discrete group, compact sets are finite. Thus $C_c(G)$ consists of finitely supported functions; for finite G, it is exactly $\mathbb{C}[G]$.

We start on $C_c(G)$ so that the main calculations involve continuous functions and compact supports.

Convolution for a Locally Compact Group

For $f, h \in C_c(G)$, define

$$\boxed{(f * h)(r) = \int_G f(s)h(s^{-1}r)\, ds.}$$

This is the finite-group coefficient formula with the sum replaced by an integral.

The product still belongs to $C_c(G)$:

$$\operatorname{supp}(f * h) \subseteq (\operatorname{supp} f)(\operatorname{supp} h),$$

and the set on the right is compact. For continuity, use

$$|(f * h)(r) - (f * h)(r_0)| \leq \|f\|_1 \sup_{s \in \operatorname{supp} f} \left| h(s^{-1}r) - h(s^{-1}r_0) \right|.$$

For $f \neq 0$, the supremum tends to zero as $r \to r_0$ by continuity and compactness. The case $f = 0$ is immediate.

Inversion May Change Haar Measure

Left invariance does not imply invariance under $s \mapsto s^{-1}$. The *modular function* is the continuous homomorphism $\Delta : G \to (0, \infty)$ specified by

$$\int_G F(ts)\, dt = \Delta(s)^{-1} \int_G F(t)\, dt.$$

We use the associated change-of-variables formula

$$\boxed{\int_G F(s^{-1})\, ds = \int_G F(t)\Delta(t^{-1})\, dt.}$$

These formulas hold for $F \in C_c(G)$; their general proofs are left to the reading.

For discrete, compact, and abelian groups, $\Delta \equiv 1$. Such groups are called *unimodular*.

Convention and inversion formula: Williams, §1.3, Lemmas 1.61 and 1.67.

The Involution Includes the Modular Factor

Define, for $f \in C_c(G)$,

$$\boxed{f^*(s) = \Delta(s^{-1})\,\overline{f(s^{-1})}.}$$

This belongs to $C_c(G)$, and the operation is conjugate-linear. Since $\Delta(s^{-1})\Delta(s) = 1$,

$$\begin{aligned}(f^*)^*(s) &= \Delta(s^{-1})\,\overline{f^*(s^{-1})} \\ &= \Delta(s^{-1})\Delta(s)f(s) = f(s).\end{aligned}$$

When $\Delta \equiv 1$, this is the finite-group formula.

The reason for the modular factor will be explicit when we take the adjoint of an integrated representation.

Integrating a Unitary Representation

Let $U : G \to \mathcal{U}(\mathcal{H})$ be strongly continuous. For $f \in C_c(G)$, we want

$$\pi_U(f)\xi = \int_G f(s)U(s)\xi \, ds.$$

We can define this using only scalar integrals. Require

$$\boxed{\langle \eta, \pi_U(f)\xi \rangle = \int_G f(s) \langle \eta, U(s)\xi \rangle \, ds \qquad (\eta, \xi \in \mathcal{H}).}$$

The scalar integrand is continuous and compactly supported. Also,

$$\left| \int_G f(s) \langle \eta, U(s)\xi \rangle \, ds \right| \leq \|f\|_1 \|\eta\| \|\xi\|.$$

Thus the proposed expression is bounded in both vectors.

The Integrated Operator Exists and Is Bounded

For fixed ξ, apply the Riesz theorem from Meeting 3 to the bounded linear functional

$$\eta \longmapsto \int_G \overline{f(s)}\, \langle U(s)\xi, \eta \rangle \, ds.$$

Taking conjugates gives a unique vector $\pi_U(f)\xi$ with the required inner products. Uniqueness shows that $\xi \mapsto \pi_U(f)\xi$ is linear.

The previous estimate and the Hilbert-space norm formula give

$$\|\pi_U(f)\xi\| \leq \|f\|_1 \, \|\xi\|\,, \qquad \boxed{\|\pi_U(f)\| \leq \|f\|_1\,.}$$

The map $f \mapsto \pi_U(f)$ is linear as well.

Integrated Operators Preserve Convolution

The integrals below are understood through their scalar matrix coefficients. The group law gives

$$\begin{aligned}\pi_U(f)\pi_U(h) &= \int_G \int_G f(s)h(t)U(st)\,dt\,ds \\ &= \int_G \int_G f(s)h(s^{-1}r)U(r)\,dr\,ds \\ &= \int_G (f * h)(r)U(r)\,dr = \pi_U(f * h).\end{aligned}$$

The second line uses $r = st$ and left invariance of Haar measure. The third uses Fubini's theorem.

The scalar double integrals are absolutely integrable, since their absolute values are bounded in integral by $\|f\|_1 \|h\|_1 \|\eta\| \|\xi\|$.

Integrated Operators Preserve the Involution

Again interpret the integrals through matrix coefficients. Unitarity and the inversion formula give

$$\begin{aligned}\pi_U(f)^* &= \int_G \overline{f(s)}\, U(s)^*\, ds \\ &= \int_G \overline{f(s)}\, U(s^{-1})\, ds \\ &= \int_G \Delta(t^{-1})\overline{f(t^{-1})}\, U(t)\, dt \\ &= \pi_U(f^*).\end{aligned}$$

Why the modular factor is necessary

It compensates for the change in Haar measure under inversion. Without it, this adjoint identity can fail on a nonunimodular group.

Example: Integrating Translations on the Real Line

Use the translation representation from Meeting 3:

$$(U(a)\xi)(x) = \xi(x-a), \qquad \xi \in L^2(\mathbb{R}).$$

Here Haar measure is da and $\Delta = 1$. For $f \in C_c(\mathbb{R})$,

$$(\pi_U(f)\xi)(x) = \int_{\mathbb{R}} f(a)\xi(x-a)\,da = (f * \xi)(x).$$

For continuous compactly supported ξ, this is an ordinary continuous function. For general $\xi \in L^2(\mathbb{R})$, it is an L^2 identity.

$$f^*(a) = \overline{f(-a)}, \qquad \|\pi_U(f)\| \leq \|f\|_1 .$$

Integrating the symmetry operators produces a bounded convolution operator.

The Left Regular Representation on $L^2(G)$

On the Hilbert space $L^2(G, \mu)$, define

$$(\lambda(s)\xi)(t) = \xi(s^{-1}t).$$

Left invariance gives

$$\|\lambda(s)\xi\|_2^2 = \int_G \left|\xi(s^{-1}t)\right|^2 \, dt = \int_G |\xi(r)|^2 \, dr = \|\xi\|_2^2 .$$

Moreover,

$$\lambda(s)\lambda(t) = \lambda(st), \qquad \lambda(s)^{-1} = \lambda(s^{-1}).$$

Thus these operators are unitary. We next check strong continuity, so that λ is among the representations used in the full norm.

Why the Regular Representation Is Strongly Continuous

First let $\xi \in C_c(G)$. As $s \to e$, continuity on compact sets gives

$$\sup_{t \in G} \left| \xi(s^{-1}t) - \xi(t) \right| \longrightarrow 0.$$

For s in a fixed compact neighborhood of e, the differences have support in a fixed compact set K. Hence

$$\|\lambda(s)\xi - \xi\|_2 \leq \mu(K)^{1/2} \sup_t \left| \xi(s^{-1}t) - \xi(t) \right| \longrightarrow 0.$$

We use the standard result that $C_c(G)$ is dense in $L^2(G)$. Given $\eta \in L^2(G)$, choose $\xi \in C_c(G)$ close to η. Then

$$\|\lambda(s)\eta - \eta\|_2 \leq 2\,\|\eta - \xi\|_2 + \|\lambda(s)\xi - \xi\|_2 .$$

This proves continuity at e; the group law gives continuity everywhere.

Density and translation continuity: see Folland, Chapters 2–3.

The Regular Representation Is Faithful on $C_c(G)$

Its integrated form acts by convolution:

$$\lambda(f)\xi = f * \xi \qquad (\xi \in C_c(G)).$$

Suppose $f \neq 0$. Choose the test vector

$$\xi(t) = \overline{f(t^{-1})} \in C_c(G) \subseteq L^2(G).$$

Then

$$(f * \xi)(e) = \int_G f(s)\xi(s^{-1})\, ds = \int_G |f(s)|^2 \, ds > 0.$$

The function $f * \xi$ is continuous. It is therefore nonzero on an open neighborhood of e, which has positive Haar measure. Thus it is *not* the zero vector in $L^2(G)$.

$$\boxed{f \neq 0 \implies \lambda(f) \neq 0.}$$

Convolution and Involution Satisfy the Algebra Laws

We have proved that λ is linear, injective, and satisfies

$$\lambda(f * h) = \lambda(f)\lambda(h), \qquad \lambda(f^*) = \lambda(f)^*.$$

The algebra laws in $\mathcal{B}(L^2(G))$ therefore give the corresponding laws on $C_c(G)$. For example,

$$\lambda((f * h) * k) = \lambda(f)\lambda(h)\lambda(k) = \lambda(f * (h * k)),$$
$$\lambda((f * h)^*) = (\lambda(f)\lambda(h))^* = \lambda(h^* * f^*).$$

Injectivity gives associativity and $(f * h)^* = h^* * f^*$.

Together with conjugate-linearity and $(f^*)^* = f$, these make $C_c(G)$ an algebra with an involution.

Defining the Full Group Norm

For $f \in C_c(G)$, define

$$\boxed{\|f\|_{\mathrm{u}} = \sup_{U} \|\pi_U(f)\|,}$$

where U ranges over all strongly continuous unitary representations of G on nonzero complex Hilbert spaces.

The bound for integrated operators gives

$$\|\lambda(f)\| \leq \|f\|_{\mathrm{u}} \leq \|f\|_1 < \infty.$$

The Full Group Norm Is a Norm

For $f, h \in C_c(G)$ and $\alpha \in \mathbb{C}$,

$$\|\alpha f\|_{\mathrm{u}} = \sup_U |\alpha| \, \|\pi_U(f)\| = |\alpha| \, \|f\|_{\mathrm{u}},$$

and

$$\begin{aligned} \|f + h\|_{\mathrm{u}} &\leq \sup_U (\|\pi_U(f)\| + \|\pi_U(h)\|) \\ &\leq \|f\|_{\mathrm{u}} + \|h\|_{\mathrm{u}}. \end{aligned}$$

The essential remaining property uses faithfulness of λ:

$$\|f\|_{\mathrm{u}} = 0 \implies \lambda(f) = 0 \implies f = 0.$$

Conversely, $\|0\|_{\mathrm{u}} = 0$. Nonnegativity is immediate, and finiteness was proved on the previous slide. All norm axioms are now verified.

The Full Norm Respects Convolution and Involution

For each U,

$$\|\pi_U(f * h)\| \leq \|\pi_U(f)\| \, \|\pi_U(h)\| \leq \|f\|_{\mathrm{u}} \, \|h\|_{\mathrm{u}} .$$

Therefore

$$\boxed{\|f * h\|_{\mathrm{u}} \leq \|f\|_{\mathrm{u}} \, \|h\|_{\mathrm{u}} .}$$

The adjoint identity also gives

$$\|f^*\|_{\mathrm{u}} = \sup_U \|\pi_U(f)^*\| = \sup_U \|\pi_U(f)\| = \|f\|_{\mathrm{u}} .$$

In particular, the involution is norm-continuous.

The Norm-Square Identity for the Full Group Norm

For every strongly continuous unitary representation U,

$$\|\pi_U(f^* * f)\| = \|\pi_U(f)^* \pi_U(f)\| = \|\pi_U(f)\|^2 .$$

Taking the supremum,

$$\begin{aligned} \|f^* * f\|_{\mathrm{u}} &= \sup_U \|\pi_U(f)\|^2 \\ &= \left(\sup_U \|\pi_U(f)\|\right)^2 = \|f\|_{\mathrm{u}}^2 . \end{aligned}$$

What has been proved so far

$C_c(G)$ has a submultiplicative norm, an isometric involution, and the norm-square identity. Completeness still has to be supplied.

Completing the Group Algebra

Let A_G be the completion of $(C_c(G), \|\cdot\|_{\mathrm{u}})$. We identify $C_c(G)$ with its dense image in A_G.

For $a, b \in A_G$, choose $f_n \to a$ and $h_n \to b$ in the full norm. Define

$$ab = \lim_n (f_n * h_n), \qquad a^* = \lim_n f_n^*.$$

These limits exist. Indeed,

$$\begin{aligned} \|f_n * h_n - f_m * h_m\|_{\mathrm{u}} &\leq \|f_n - f_m\|_{\mathrm{u}} \|h_n\|_{\mathrm{u}} \\ &\quad + \|f_m\|_{\mathrm{u}} \|h_n - h_m\|_{\mathrm{u}}, \\ \|f_n^* - f_m^*\|_{\mathrm{u}} &= \|f_n - f_m\|_{\mathrm{u}}. \end{aligned}$$

Cauchy sequences are bounded, so both right-hand sides tend to zero. The same estimates show independence of the chosen approximating sequences.

The Identities Pass to the Completion

The estimates imply that multiplication is jointly continuous and the involution is isometric. Hence the algebraic identities on $C_c(G)$ pass to A_G.

$$\|ab\|_{\mathrm{u}} \leq \|a\|_{\mathrm{u}} \|b\|_{\mathrm{u}}\,, \qquad \|a^*\|_{\mathrm{u}} = \|a\|_{\mathrm{u}}\,.$$

If $f_n \to a$ in the full norm, then $f_n^* * f_n \to a^* a$. Therefore

$$\begin{aligned} \|a^* a\|_{\mathrm{u}} &= \lim_n \|f_n^* * f_n\|_{\mathrm{u}} \\ &= \lim_n \|f_n\|_{\mathrm{u}}^2 = \|a\|_{\mathrm{u}}^2\,. \end{aligned}$$

Thus A_G is complete and has the same norm and involution identities as $\mathcal{B}(\mathcal{H})$. A multiplicative identity has not been assumed.

Every Integrated Representation Extends to the Completion

The definition of the full norm gives

$$\|\pi_U(f)\| \leq \|f\|_{\mathrm{u}} \qquad (f \in C_c(G)).$$

Since $\mathcal{B}(\mathcal{H})$ is complete, there is a unique continuous extension

$$\widetilde{\pi}_U : A_G \to \mathcal{B}(\mathcal{H}), \qquad \widetilde{\pi}_U(a) = \lim_n \pi_U(f_n) \quad \text{when } f_n \to a.$$

Continuity preserves the identities:

$$\widetilde{\pi}_U(ab) = \widetilde{\pi}_U(a)\widetilde{\pi}_U(b), \qquad \widetilde{\pi}_U(a^*) = \widetilde{\pi}_U(a)^*.$$

Thus completing the algebra does not lose any of the representations used to define its norm.

Section 4

The Definition of a C^*-Algebra

Algebras with an Involution

A *complex algebra* is a complex vector space with an associative, bilinear multiplication $(a, b) \mapsto ab$.

An *involution* is a map $a \mapsto a^*$ satisfying

$$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*,$$
$$(ab)^* = b^* a^*, \qquad (a^*)^* = a.$$

A complex algebra with an involution is called a $*$-*algebra*.

Our two constructions

In $\mathcal{B}(\mathcal{H})$, multiplication is composition and the involution is the adjoint. In $C_c(G)$, multiplication is convolution and the involution contains the modular factor.

Definition of a C^*-Algebra

Definition

A *C^*-algebra* is a complex $*$-algebra A with a norm such that A is complete and, for all $a, b \in A$,

$$\|ab\| \leq \|a\| \, \|b\|, \qquad \boxed{\|a^*a\| = \|a\|^2.}$$

The boxed equality is the *C^*-identity*.

A multiplicative identity is not required. If one exists, the algebra is called *unital*. Commutativity is not required either.

The involution is automatically isometric: for $a \neq 0$,

$$\|a\|^2 = \|a^*a\| \leq \|a^*\| \, \|a\| \quad \Longrightarrow \quad \|a\| \leq \|a^*\|.$$

Apply the same argument to a^* to obtain equality.

Bounded Operators and Full Group C^*-Algebras

We have proved, that

$$\boxed{\mathcal{B}(\mathcal{H}) \text{ is a unital } C^*\text{-algebra.}}$$

We have also proved that A_G is a C^*-algebra. It is called the *full group C^*-algebra* and is denoted by

$$\boxed{C^*(G).}$$

It is unital if and only if G is discrete.

References for Further Reading

Hilbert-space operators and group algebras

Ian F. Putnam, *C^*-algebras* (lecture notes, 2013): §§1.1–1.2 for definitions and operator examples; §§2.1–2.3 for finite groups; §2.8 for locally compact groups.

The analytic details behind the locally compact case

Gerald B. Folland, *A Course in Abstract Harmonic Analysis*, 2nd edition: Chapter 2 for Haar measure, the modular function, and convolution; §3.2 for integrated representations; §7.1 for the group C^*-algebra.