CMSP / Operator Algebras as Physical Systems

Meeting 3:

Banach Spaces · Hilbert Spaces · Unitary Group Representations

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Section 1

Banach Spaces

Hilbert Spaces

Unitary Group Representations

From Meeting 2 to Chapter 2

- A metric lets us discuss *convergence*, *Cauchy sequences*, *completeness* and *compactness*.
- A vector space lets us add vectors and multiply them by scalars.
- A *norm* relates these structures:

$$d(x, y) = \|x - y\| .$$

- Kreyszig, Chapter 2: the central objects are *normed spaces*, *Banach spaces*, *linear operators* and *linear functionals*.

Normed Spaces and Banach Spaces (§2.2)

Definition

A *normed space* is a vector space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, equipped with $\|\cdot\|$ such that

$$\|x\| = 0 \iff x = 0, \qquad \|\alpha x\| = |\alpha| \, \|x\|, \qquad \|x + y\| \leq \|x\| + \|y\| .$$

A *Banach space* is a normed space complete in $d(x, y) = \|x - y\|$.

- Norm convergence: $x_n \to x \iff \|x_n - x\| \to 0$.
- Every Banach space is a complete metric space, but its norm also encodes linear structure.

Canonical Banach Spaces (§2.2)

- $\mathbb{C}^n$: $\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}$.
- ℓ^p, $1 \le p < \infty$: $\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}$.
- ℓ^∞: $\|x\|_\infty = \sup_{j \ge 1} |x_j|$.
- $C[a,b]$: $\|f\|_\infty = \max_{t \in [a,b]} |f(t)|$.
- $L^p[a,b]$: obtained, for example, as the completion of $C[a,b]$ with the L^p norm (functions are identified almost everywhere).

A useful distinction

$C[a,b]$ is complete in the *supremum* norm, but generally not in the L^2 norm.

Closed Subspaces and Completion (§2.3)

Subspace

If X is Banach and $Y \subseteq X$ is a linear subspace, then

$$Y \text{ is Banach} \iff Y \text{ is closed in } X.$$

Completion

Every normed space X admits a Banach completion $\widehat{X}$, with a norm-preserving linear embedding

$$J : X \longrightarrow \widehat{X}, \qquad \overline{J(X)} = \widehat{X}.$$

- The passage from continuous functions with an L^2 norm to L^2 is a motivating example.

Infinite Series and Schauder Bases (§2.3)

Completeness controls sums

For a normed space X,

$$X \text{ is Banach} \iff \sum_{n=1}^{\infty} \|x_n\| < \infty \implies \sum_{n=1}^{\infty} x_n \text{ converges in } X.$$

- A *Schauder basis* (e_n) allows a unique norm-convergent expansion

$$x = \sum_{n=1}^{\infty} a_n e_n.$$

- In ℓ^p for $1 \le p < \infty$, the coordinate vectors e_n form a Schauder basis.
- In infinite dimensions, a Schauder basis is not the same as an algebraic (Hamel) basis.

Finite Dimension Is Special (§2.4)

Three fundamental facts

Every finite-dimensional normed space is complete; every finite-dimensional subspace of a normed space is closed; and every two norms on a fixed finite-dimensional space are equivalent.

$$c\,\|x\|_a \le \|x\|_b \le C\,\|x\|_a \qquad (c, C > 0).$$

- Thus, in finite dimensions, changing norms does not change convergence.
- Infinite-dimensional spaces do *not* share these simplifications in general.

Compactness and Dimension (§2.5)

Heine–Borel in finite dimensions

For a finite-dimensional normed space X and $K \subseteq X$,

$$K \text{ compact} \quad \Longleftrightarrow \quad K \text{ closed and bounded.}$$

Riesz's lemma: the obstruction

An infinite-dimensional normed space has a noncompact closed unit ball.

For the standard vectors (e_n) in ℓ^2,

$$\|e_n\| = 1, \qquad \|e_n - e_m\| = \sqrt{2} \quad (n \neq m).$$

There is no convergent subsequence: *boundedness alone is not enough*.

Linear Operators (§2.6)

Definition

A map $T : X \to Y$ is linear if

$$T(\alpha x + \beta y) = \alpha Tx + \beta Ty.$$

- Ker $T = \{x \in X : Tx = 0\}$ and Ran $T = \{Tx : x \in X\}$ are vector subspaces.
- In finite dimensions, choosing bases turns T into a matrix.
- In function spaces, important operators include integration, differentiation and multiplication.
- Linearity alone does *not* guarantee continuity in infinite dimensions.

Bounded $\iff$ Continuous (§2.7)

Fundamental theorem

A linear operator $T : X \to Y$ between normed spaces is continuous if and only if it is bounded:

$$\exists C \geq 0 \ \forall x \in X : \quad \|Tx\| \leq C \|x\| .$$

- For bounded T, $\|Tx - Ty\| \leq \|T\| \, \|x - y\|$.
- The norm of T is
$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$
- An unbounded linear operator (such as differentiation on a suitable normed space) may still be physically important.

Operator Norm and Composition (§§2.7–2.9)

For bounded linear operators $T : X \to Y$ and $S : Y \to Z$,

$$\|ST\| \leq \|S\| \, \|T\|, \qquad \|T^n\| \leq \|T\|^n \quad (T : X \to X).$$

- The operator norm makes limits of operators meaningful.
- In finite dimensions, every linear operator is bounded.
- In infinite dimensions, we must distinguish *bounded* from *(densely defined) unbounded* operators.

Why this matters later

Unitary transformations will be a particularly rigid class of bounded, invertible operators on a Hilbert space.

Linear Functionals and the Dual (§§2.8, 2.10)

- A *linear functional* is a linear map $f : X \to \mathbb{K}$.
- The *continuous dual* X' is the space of bounded linear functionals, with

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)| .$$

Dual spaces are complete

X' is Banach even if X is not Banach.

- Examples from Kreyszig: $(\ell^1)' \cong \ell^\infty$ and $(\ell^p)' \cong \ell^q$ for $1 < p < \infty$, where $1/p + 1/q = 1$.
- In a Hilbert space, *every* bounded linear functional has a simple representation by an inner product.

Bounded Operators Form a Banach Space (§2.10)

Theorem

If Y is Banach, the normed space $\mathcal{B}(X, Y)$ of all bounded linear maps $X \to Y$ is Banach, regardless of whether X is complete.

$$\mathcal{B}(X) = \mathcal{B}(X, X), \qquad (ST)x = S(Tx), \qquad \|ST\| \leq \|S\| \, \|T\| .$$

- In the case of operators on Hilbert spaces, the adjoint T^* adds the extra structure.

Section 2

Banach Spaces

Hilbert Spaces

Unitary Group Representations

Inner Products and Hilbert Spaces

Definition

A *complex inner product* on $\mathcal{H}$ satisfies

$$\begin{aligned} \langle x, \alpha y + \beta z \rangle &= \alpha \langle x, y \rangle + \beta \langle x, z \rangle , \\ \langle x, y \rangle &= \overline{\langle y, x \rangle}, \\ \langle x, x \rangle &\geq 0, \quad \langle x, x \rangle = 0 \iff x = 0. \end{aligned}$$

A *Hilbert space* is complete in $\|x\| = \sqrt{\langle x, x \rangle}$.

Hilbert Spaces: Basic Examples

- $\mathbb{C}^n$: $\langle x, y \rangle = \sum_{j=1}^{n} \overline{x_j} y_j$.
- ℓ^2: $\langle x, y \rangle = \sum_{j=1}^{\infty} \overline{x_j} y_j$.
- $L^2(\mathbb{R}^d)$: $\langle f, g \rangle = \int_{\mathbb{R}^d} \overline{f(x)} g(x)\, dx$.

Not every Banach space is Hilbert

An inner-product norm obeys the parallelogram identity:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

For the usual norms, ℓ^p with $p \neq 2$ is not a Hilbert space.

Cauchy–Schwarz and Orthogonality

Cauchy–Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| \, .$$

- Orthogonality: $x \perp y \iff \langle x, y\rangle = 0$.
- Pythagoras: if $x \perp y$, then
$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 \, .$$
- $|\langle x, y\rangle|^2$ is the quantity appearing in quantum transition probabilities for normalized states.

Orthogonal Complements and Projections

For a closed linear subspace $M \subseteq \mathcal{H}$, define

$$M^{\perp} = \{x \in \mathcal{H} : \langle m, x \rangle = 0 \text{ for all } m \in M\}.$$

Orthogonal projection theorem

Every $x \in \mathcal{H}$ decomposes uniquely as

$$x = m + n, \qquad m \in M, \quad n \in M^{\perp}.$$

Thus $\mathcal{H} = M \oplus M^{\perp}$, with $P_M x = m$.

- $P_M^2 = P_M = P_M^*$: orthogonal projections are self-adjoint idempotents.
- In quantum theory, projections represent ideal yes/no measurements.

Orthonormal Bases and Fourier Expansion

An orthonormal family (e_j) satisfies $\langle e_j, e_k \rangle = \delta_{jk}$.

Complete orthonormal system

If $(e_j)_{j \geq 1}$ is an orthonormal basis of a separable Hilbert space, then

$$x = \sum_{j=1}^{\infty} \langle e_j, x \rangle \; e_j, \qquad \|x\|^2 = \sum_{j=1}^{\infty} |\langle e_j, x \rangle|^2 .$$

The second equality is *Parseval's identity*.

- Coefficients are amplitudes relative to an orthonormal basis.
- A unitary operator sends an orthonormal basis to another orthonormal basis.

The Riesz Representation Theorem

Theorem

For each bounded linear functional $f : \mathcal{H} \to \mathbb{C}$, there is a unique $v \in \mathcal{H}$ such that

$$f(x) = \langle v, x \rangle \quad (x \in \mathcal{H}), \qquad \|f\| = \|v\| .$$

- The continuous dual of a Hilbert space is represented by vectors of that same space (with the appropriate complex conjugation convention).
- This allows us to define the adjoint T^* of a bounded operator T through an inner-product identity.

Hilbert Adjoint and Operator Classes

For $T \in \mathcal{B}(\mathcal{H})$, the unique adjoint T^* is characterized by

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \qquad (x, y \in \mathcal{H}).$$

Three important classes

- **Self-adjoint:** $T^* = T$.
- **Normal:** $T^*T = TT^*$.
- **Unitary:** $T^*T = TT^* = I$.

In finite dimensions: T^* is conjugate transpose, and unitary means $U^\dagger U = UU^\dagger = I$.

Unitary Operators: The Key Properties

Equivalent characterizations

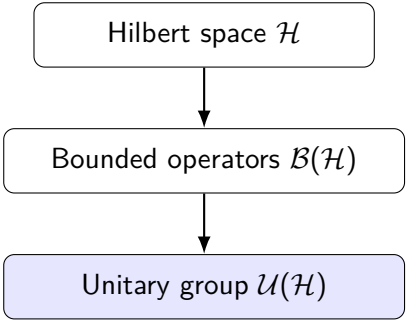
A bounded linear map $U : \mathcal{H} \to \mathcal{H}$ is unitary if and only if it is surjective and inner-product preserving:

$$\langle Ux, Uy \rangle = \langle x, y \rangle \qquad (x, y \in \mathcal{H}).$$

Equivalently, $U^{-1} = U^*$.

- $\|Ux\| = \|x\|$: probabilities and distances are preserved.
- Products and inverses of unitary operators are unitary.
- *Surjectivity is essential*: the unilateral shift on ℓ^2 preserves norms but is not unitary.

From Hilbert Spaces to Operator Algebras



Key algebraic identities

$\mathcal{B}(\mathcal{H})$ is a C^*-algebra: $\|T^*T\| = \|T\|^2$.

$\mathcal{U}(\mathcal{H}) = \{U \in \mathcal{B}(\mathcal{H}) : U^*U = UU^* = I\}$ is a group under composition.

Section 3

Banach Spaces

Hilbert Spaces

Unitary Group Representations

Why Do We Study Symmetry in Theoretical Physics?

- **Rotations** organize angular momentum and spin.
- **Spatial translations** organize momentum.
- **Time translations** organize energy for time-independent systems.
- **Internal symmetries** organize conserved charges and particle multiplets.

The central question

How can a physical symmetry g act on quantum states without changing transition probabilities?

$$\text{symmetry group } G \quad \longrightarrow \quad \text{operators on the state Hilbert space } \mathcal{H}.$$

Quantum States Are Rays, Not Individual Vectors

A normalized vector $|\psi\rangle \in \mathcal{H}$ represents the same pure state as $e^{i\alpha}|\psi\rangle$.

$$|\psi\rangle \sim e^{i\alpha}|\psi\rangle, \qquad \left|e^{i\alpha}\right| = 1.$$

Born transition probabilities

For normalized $|\psi\rangle, |\phi\rangle$, the transition probability is

$$P(\phi \mid \psi) = |\langle\phi \mid \psi\rangle|^2 .$$

A physical symmetry must preserve such probabilities, not necessarily the phase of every vector.

Wigner's Theorem and an Important Subtlety

Wigner's theorem (informal form)

A bijective symmetry of pure-state rays preserving transition probabilities can be implemented by a *unitary or antiunitary* transformation, unique up to a phase.

- Continuously connected symmetries are represented locally by unitary operators.
- Time reversal provides an important antiunitary example.
- Since quantum states are rays, the group law may hold only *up to a phase*:

$$U(g)U(h) = e^{i\omega(g,h)}U(gh).$$

This is a *projective unitary representation*; ordinary representations have $e^{i\omega(g,h)} = 1$.

Groups and Unitary Representations

Definition

A *unitary representation* of a group G on $\mathcal{H}$ is a homomorphism

$$U : G \longrightarrow \mathcal{U}(\mathcal{H}), \qquad U(gh) = U(g)U(h).$$

Therefore

$$U(e) = I, \qquad U(g^{-1}) = U(g)^{-1} = U(g)^{*}.$$

- For a topological group, we usually also require *strong continuity*: $g \mapsto U(g)\psi$ is norm-continuous for every $\psi \in \mathcal{H}$.
- The abstract group law becomes composition of physical transformations.

What Unitary Representations Preserve

For all $g \in G$ and $\psi, \phi \in \mathcal{H}$,

$$\langle U(g)\phi, U(g)\psi \rangle = \langle \phi, \psi \rangle , \qquad \|U(g)\psi\| = \|\psi\| .$$

State and observable transformations

$$|\psi\rangle \longmapsto U(g)|\psi\rangle, \qquad A \longmapsto U(g)AU(g)^{-1}.$$

- Inner-product preservation guarantees transition-probability preservation.
- Conjugation preserves products, adjoints and (where defined) spectra.
- Invariance of an observable is expressed by $U(g)AU(g)^{-1} = A$.

Invariant Subspaces and Reducibility

A closed subspace $M \subseteq \mathcal{H}$ is *G-invariant* if

$$U(g)M \subseteq M \qquad \text{for every } g \in G.$$

Irreducible representation

U is *irreducible* if the only closed invariant subspaces are $\{0\}$ and $\mathcal{H}$.

- For a unitary representation, M invariant implies $M^{\perp}$ invariant.
- Hence $\mathcal{H} = M \oplus M^{\perp}$ decomposes the representation into independent sectors.
- Irreducible representations are the basic building blocks for symmetry classifications.

Schur's Lemma: A Useful Consequence

Finite-dimensional complex version

If $U : G \to \mathcal{U}(\mathcal{H})$ is irreducible and $\dim \mathcal{H} < \infty$, every linear operator A commuting with all $U(g)$ is scalar:

$$AU(g) = U(g)A \quad \forall g \in G \quad \Longrightarrow \quad A = \lambda I.$$

- Idea: an eigenvalue of A exists over $\mathbb{C}$; its eigenspace is invariant, so irreducibility forces it to be all of $\mathcal{H}$.
- Consequence: an invariant Hamiltonian restricted to a finite-dimensional irreducible sector is proportional to the identity.

A First Example: Parity as $\mathbb{Z}_2$

Let $\mathbb{Z}_2 = \{e, p\}$ with $p^2 = e$. On $\mathbb{C}^2$ define

$$U(e) = I, \qquad U(p) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- $U(p)^* U(p) = I$ and $U(p)^2 = I$.
- $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$: the representation is reducible.
- The two invariant subspaces carry the even and odd one-dimensional representations.

Physical interpretation

Parity eigenvalues $+1$ and -1 label symmetry sectors.

A Compact Example: $U(1)$ and Charges

The group $U(1) = \{e^{i\theta} : \theta \in \mathbb{R}\}$ has irreducible continuous unitary representations

$$\chi_n(e^{i\theta}) = e^{in\theta}, \qquad n \in \mathbb{Z}.$$

- These are one-dimensional characters.
- Integer n follows from requiring 2π-periodicity: $\chi_n(e^{i(\theta+2\pi)}) = \chi_n(e^{i\theta})$.
- More generally, $U(1)$ acts on a Hilbert space as a direct sum of charge sectors.

Interpretation

Symmetry labels are representation-theoretic data. The units and normalization of physical charge must be specified separately.

Continuous Symmetries and Stone's Theorem

A strongly continuous one-parameter unitary group satisfies

$$U(t+s) = U(t)U(s), \qquad U(0) = I.$$

Stone's theorem

There is a unique self-adjoint operator A (possibly unbounded) such that

$$U(t) = e^{-itA}.$$

With physical units one often writes

$$U(t) = e^{-itQ/\hbar},$$

where Q is the generator: momentum, angular momentum or Hamiltonian, depending on the symmetry.

Spatial Translations Generate Momentum

Take $\mathcal{H} = L^2(\mathbb{R})$ and $G = (\mathbb{R}, +)$. Define

$$(U(a)\psi)(x) = \psi(x - a).$$

- $U(a + b) = U(a)U(b)$.
- Substitution in the integral gives $\|U(a)\psi\|_2 = \|\psi\|_2$.
- On a suitable domain, the generator is

$$U(a) = e^{-iaP/\hbar}, \qquad P = -i\hbar \frac{d}{dx}.$$

Physics

Momentum arises as the self-adjoint generator of spatial translations.

Time Translations Generate Energy

For a time-independent self-adjoint Hamiltonian H,

$$U(t) = e^{-itH/\hbar}, \qquad |\psi(t)\rangle = U(t)|\psi(0)\rangle.$$

- Unitarity guarantees conservation of total probability.
- For vectors in the domain of H,

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle.$$

- Thus the Schrödinger equation is the infinitesimal form of a unitary one-parameter evolution.

Rotations and the Angular Momentum Algebra

For spatial rotations we seek operators implementing the rotation group.

Generators of rotations

$$U_i(\theta) = e^{-i\theta J_i/\hbar}, \qquad [J_i, J_j] = i\hbar \sum_k \varepsilon_{ijk} J_k.$$

- The J_i are self-adjoint angular-momentum generators.
- They cannot all be diagonalized simultaneously because they do not commute.
- Irreducible spin-j representations have dimension $2j+1$ and satisfy

$$J^2 = \hbar^2 j(j+1) I, \qquad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots.$$

Spin $\frac{1}{2}$: An Explicit Unitary Representation

On $\mathbb{C}^2$, let $J_k = \frac{\hbar}{2}\sigma_k$, with Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A rotation about the z-axis acts by

$$U_z(\theta) = e^{-i\theta\sigma_z/2} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}.$$

$U_z(\theta)$ is unitary and preserves every spinor's norm.

A striking fact

In the spin-$\frac{1}{2}$ representation,

$$U_z(2\pi) = -I, \qquad U_z(4\pi) = I.$$

- An ordinary 2π spatial rotation is the identity in $SO(3)$.
- But it acts as $-I$ on spinors: this changes vectors, not physical rays.
- The natural ordinary representation is of $SU(2)$, the double cover of $SO(3)$; it induces a projective representation of $SO(3)$.

$$SU(2) \longrightarrow SO(3), \qquad \{I, -I\} \longmapsto \{e\}.$$

Symmetries, Conserved Quantities and Sectors

Let $U(g)$ represent a symmetry of a time-independent Hamiltonian H.

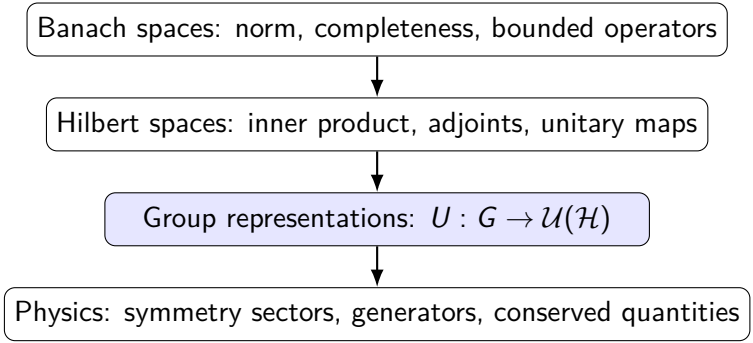
Invariance

$$U(g)HU(g)^{-1} = H \quad \Longleftrightarrow \quad [U(g), H] = 0,$$

with the appropriate domain condition if H is unbounded.

- Symmetry-invariant sectors can be chosen independently.
- For a differentiable one-parameter symmetry $U(t) = e^{-itQ/\hbar}$, invariance formally yields $[Q, H] = 0$ on a suitable common domain.
- This connects group representations to conservation laws, degeneracies and selection rules.

Putting the Pieces Together



Towards operator algebras

Group representations place a family of physically meaningful operators inside $\mathcal{B}(\mathcal{H})$, which is the starting point for studying algebras generated by symmetries and observables.