CMSP / Operator Algebras as Physical Systems

Meeting 2:

Completeness · Convergence · Compactness

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Section 1

Completeness

Convergence

Compactness

Recap: Metric Spaces

A metric space is a pair (X, d) with $d : X \times X \to [0, \infty)$ satisfying

$$d(x, y) = 0 \iff x = y, \quad d(x, y) = d(y, x), \quad d(x, z) \leq d(x, y) + d(y, z).$$

- $\mathbb{C}^n$: $d(x, y) = \left(\sum_{j=1}^{n} |x_j - y_j|^2 \right)^{1/2}$.
- ℓ^p $(1 \leq p < \infty)$: $d(x, y) = \left(\sum_{j=1}^{\infty} |x_j - y_j|^p \right)^{1/p}$.
- ℓ^∞: $d(x, y) = \sup_j |x_j - y_j|$.
- $C[a, b]$: $d(f, g) = \max_{t \in [a,b]} |f(t) - g(t)|$.

Recap: Topology and Continuity

- $B_r(x) = \{y \in X : d(x,y) < r\}$; $U \subseteq X$ is open iff each $x \in U$ has some $B_r(x) \subseteq U$.
- $F \subseteq X$ is closed iff $X \setminus F$ is open; $\overline{A}$ is the smallest closed set containing A.
- D is dense in X iff $\overline{D} = X$; X is separable iff it has a countable dense subset.

Recap: Convergence and Completeness

- $x_n \to x$ iff $d(x_n, x) \to 0$; limits are unique.
- (x_n) is Cauchy iff $\forall \varepsilon > 0 \ \exists N \ \forall m, n \geq N : \ d(x_m, x_n) < \varepsilon$.
- X is complete iff every Cauchy sequence converges to a point of X.
- If X is complete, a subspace $A \subseteq X$ is complete iff A is closed in X.
- Complete examples: $\mathbb{C}^n$, ℓ^p $(1 \leq p < \infty)$, ℓ^∞, $C[a, b]$; incomplete examples: $\mathbb{Q}$, $(0, 1)$.

Recap: Completion of Metric Spaces

Completion theorem

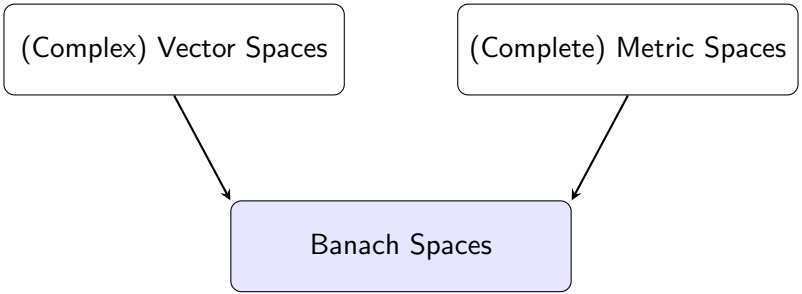
Every metric space (X, d) embeds isometrically as a dense subspace of a complete metric space $(\widehat{X}, \widehat{d})$, unique up to isometry.

$$\widehat{X} = \{\text{Cauchy sequences in } X\}/\sim, \qquad (x_n) \sim (y_n) \iff d(x_n, y_n) \to 0.$$

$$\widehat{d}([(x_n)], [(y_n)]) = \lim_{n\to\infty} d(x_n, y_n).$$

- Canonical embedding: $x \mapsto [(x, x, \ldots)]$.
- Fundamental example: $\mathbb{Q} \hookrightarrow \mathbb{R}$.

Banach Spaces



Banach Spaces: Definition

Definition

A *Banach space* is a vector space X equipped with a norm $\|\cdot\| : X \to [0, \infty)$ satisfying, for all $x, y \in X$ and $\lambda \in \mathbb{C}$,

$$\|x\| = 0 \iff x = 0, \qquad \|\lambda x\| = |\lambda|\|x\|, \qquad \|x + y\| \leq \|x\| + \|y\|,$$

such that X is complete under the induced metric $d(x, y) = \|x - y\|$.

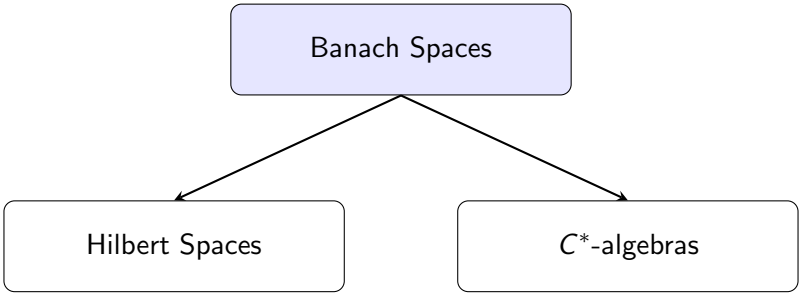
Examples of Banach Spaces

- $\mathbb{C}^n$, with $\|x\|_2 = \left(\sum_{j=1}^{n} |x_j|^2\right)^{1/2}$.
- ℓ^p $(1 \leq p < \infty)$, with $\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p}$.
- ℓ^∞, with $\|x\|_\infty = \sup_j |x_j|$.
- $C([a,b])$, with $\|f\|_\infty = \max_{t\in[a,b]} |f(t)|$.
- $\mathcal{B}(X)$: bounded linear operators on a Banach space X, with operator norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$.

More Examples of Banach Spaces

- $L^p(\mathbb{R}^d)$ $(1 \leq p < \infty)$: $\|f\|_p = \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p}$.
- $W^{k,p}(\Omega)$ (Sobolev spaces): $\|f\|_{W^{k,p}} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p}$, where D^α denotes weak derivative.
- $\mathcal{S}_1(H)$ (trace-class operators): $\|T\|_1 = \mathrm{Tr}(|T|)$. (Density operators of quantum mechanics belong to this space.)

Banach Spaces



Section 2

Completeness

Convergence

Compactness

Convergence Spaces

Informal Definition

A *convergence space* is a space in which we can talk about convergence.

- A *net* in X is a family $(x_\alpha)_{\alpha \in D}$ indexed by a directed set D.
- A directed set is a preordered set in which any two elements have a common upper bound.
- Nets generalize sequences. We write $x_\alpha \to x$ when the net converges to x.

Continuity

A map $f : X \to Y$ is *continuous* if it preserves convergence:

$$x_\alpha \to x \implies f(x_\alpha) \to f(x).$$

Hausdorff Convergence Spaces

Definition

A convergence space X is *Hausdorff* if every convergent net has a unique limit:

$$x_\alpha \to x, \quad x_\alpha \to y \implies x = y.$$

Topological Convergence Spaces

Definition

A *topological space* is a pair (X, τ), where $\tau \subseteq \mathcal{P}(X)$ satisfies:

- $\emptyset, X \in \tau$.
- Arbitrary unions of elements of τ belong to τ.
- Finite intersections of elements of τ belong to τ.

A convergence space X is *topological* if there exists a topology τ on X such that, for every net $(x_\alpha)_{\alpha \in D}$,

$$x_\alpha \to x \iff \forall U \in \tau \text{ with } x \in U, \quad x_\alpha \in U \text{ eventually.}$$

Here, *eventually* means

$$\exists \alpha_0 \in D \quad \forall \alpha \geq \alpha_0 : \; x_\alpha \in U.$$

Metric Spaces as Convergence Spaces

Proposition

Every metric space is a Hausdorff topological convergence space.

Let (X, d) be a metric space. The metric induces a topology

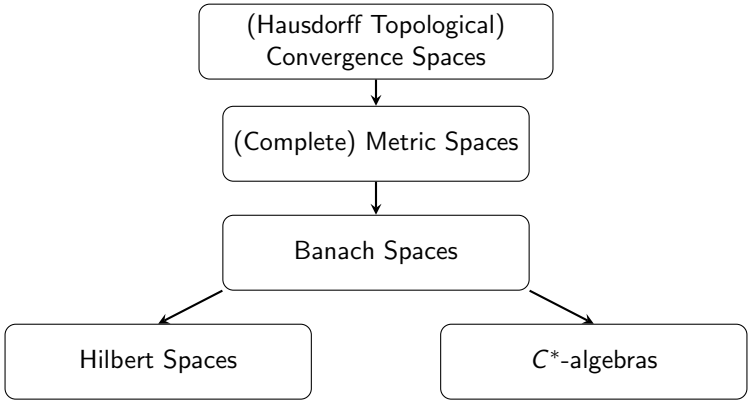
$$\tau_d = \{U \subseteq X : \forall x \in U,\ \exists \varepsilon > 0,\ B_\varepsilon(x) \subseteq U\}.$$

Convergence of nets is characterized by

$$x_\alpha \to x \iff \forall \varepsilon > 0,\ d(x_\alpha, x) < \varepsilon \text{ eventually.}$$

- **Topological:** Convergence is determined by the open balls.
- **Hausdorff:** If $x \neq y$, then $B_{d(x,y)/3}(x)$ and $B_{d(x,y)/3}(y)$ are disjoint.

Convergence Spaces



Section 3

Completeness

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Compactness

Compactness in Convergence Spaces

Definition via Nets

A convergence space X is *compact* if every net in X admits a subnet converging to a point of X:

$$\forall (x_\alpha)_{\alpha \in D} \quad \exists (x_{\alpha(\beta)})_{\beta \in E},\ x \in X :\ x_{\alpha(\beta)} \to x.$$

Equivalent Characterization via Filters

A *filter* on X is a nonempty family of subsets, closed under finite intersections and supersets, not containing $\emptyset$.

$$X \text{ compact} \iff \text{every ultrafilter on } X \text{ converges.}$$

Compactness in Topological Convergence Spaces

Open-Cover Characterization

For a topological space (X, τ), compactness is equivalent to every open cover admitting a finite subcover:

$$X = \bigcup_{i \in I} U_i, \ U_i \in \tau \implies X = \bigcup_{k=1}^{n} U_{i_k}.$$

Closed-Set Characterization

Equivalently, every family $(F_i)_{i \in I}$ of closed subsets with the *finite intersection property* has nonempty intersection:

$$\left(\bigcap_{i \in J} F_i \neq \emptyset \quad \forall J \subseteq I \text{ finite} \right) \implies \bigcap_{i \in I} F_i \neq \emptyset.$$

Compactness in Metric Spaces

Sequential Compactness

For $K \subseteq (X, d)$, compactness is equivalent to every sequence in K having a subsequence converging to a point of K:

$$(x_n) \subseteq K \implies \exists (x_{n_j}),\ x \in K :\ d(x_{n_j}, x) \to 0.$$

Total Boundedness

K is *totally bounded* if, for every $\varepsilon > 0$, finitely many ε-balls cover K:

$$\forall \varepsilon > 0\ \exists x_1, \ldots, x_N \in K :\ K \subseteq \bigcup_{j=1}^{N} B_\varepsilon(x_j).$$

K compact $\iff$ K complete and totally bounded.

Compactness in Finite-Dimensional Vector Spaces

Heine–Borel Theorem

Let V be a finite-dimensional Banach space and $K \subseteq V$. Then

$$K \text{ compact} \iff K \text{ closed and bounded.}$$

Locally Compact Spaces

Definition

A (Hausdorff topological) convergence space X is *locally compact* if every point has a compact neighborhood:

$$\forall x \in X \ \exists U \in \tau,\ K \subseteq X : \quad x \in U \subseteq K, \quad K \text{ compact.}$$

- Every compact space is locally compact.

Examples of Locally Compact Spaces

- $\mathbb{R}^n$ and $\mathbb{C}^n$ with their usual convergences.
- Compact spaces: $[0,1]$, S^n, $\mathbb{T}^n$, the Cantor set.
- Hausdorff topological manifolds and Lie groups, such as $\mathrm{GL}_n(\mathbb{C})$.
- Any discrete space, including infinite discrete spaces.
- The p-adic field $\mathbb{Q}_p$: each $a + p^k\mathbb{Z}_p$ is compact and open.

Nonexample

An infinite-dimensional convergence vector space is never locally compact.

Our Playground

